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ABSTRACT

This document is intended as a resource for persons using, designing, or evaluating instructional materials in whole number subtraction. Its purpose is to provide conceptual machinery: (1) for describing/specifying subtraction tests and exercises and (2) for formulating related questions and conjectures. It is mainly a logical analysis subject to revision on the basis of student data, information from the literature, and the criticism of peers. (Author)

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Computer-based Education
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WHOLE NUMBER SUBTRACTION
AN ANALYSIS

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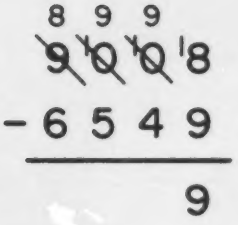

WHOLE NUMBER SUBTRACTION -- AN ANALYSIS

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January 1980

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INTRODUCTION

This document is intended as a resource for persons using, designing, or evaluating instructional materials in whole number subtraction. Its purpose is to provide conceptual machinery (a) for describing/specifying subtraction tests and exercises and (b) for formulating related questions and conjectures. It is mainly a logical analysis subject to revision on the basis of student data, information from the literature, and the criticism of peers.

SUBTRACTION PROBLEMS

A subtraction problem is considered to consist of two "whole numerals" and a minus sign, where these three elements are configured in the form conventional for hand computation, and where the upper numeral (the "minuend") has a value greater than or equal to that of the lower numeral (the "subtrahend").

"UNITS"

Here is a routine subtraction problem.

A. $\begin{array}{r} 130342 \\ -\ 31333 \\ \hline \end{array}$

Notice that this problem can be broken into parts, each of which can stand alone as a valid subtraction problem--

B. $\begin{array}{r} 130 \\ -\ 31 \\ \hline \end{array}$ C. $\begin{array}{r} 3 \\ -3 \\ \hline \end{array}$ D. $\begin{array}{r} 42 \\ -33 \\ \hline \end{array}$

However, none of these subsidiary problems can be broken into further parts all of which can stand alone. Problems which, like these three, cannot be partitioned exhaustively into simpler problems, will be called "units". Every subtraction problem is made up of units (one or more).

The difficulties students experience in subtraction problems fall into two classes, (a) those that occur even when a unit stands alone, and (b) those that may arise when a unit appears together with one or more other units in a larger problem.

"PAIRS"

Every subtraction problem can be analyzed into a sequence of vertical pairs of digits ordered from the right. For example, in problem A above, the first pair is $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$, the second is $\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}$, ..., the sixth is $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$.*

*Leading blanks in a subtrahend are a kind of "abbreviation" for the leading zeros which mathematically stand there. For purposes of analysis, we sometimes will write the zeros out explicitly.

2 The rightmost pair of digits in any subtraction problem must be one of the one hundred possible pairs displayed in Fig. 1. If it is one of the 55 pairs in the upper right-hand portion of the diagram, it is, in itself, a unit (for instance, example C above). If it is one of the 45 pairs in the lower left-hand portion of the diagram, it is part of a larger unit.

In the latter case, the pair next to the left is examined. If it is one of the 45 pairs in the upper right-hand portion of Fig. 2, the two pairs together constitute a unit (for instance, example D above). (Since computation proceeds from right to left, we say that the pair on the right "initiates" the unit, while the one on the left "terminates" the unit.)

Alternatively, if the pair on the left is one of the 55 in the lower left-hand portion of Fig. 2, it "continues" the unit but does not terminate it (as in example B above), and one must examine the next pair to the left. This process continues until a terminating pair (one of the 45 in the upper right-hand portion of Fig. 2) is found.

In general, then, a unit is made up of three kinds of pairs: an "initiator" on the right, a "terminator" on the left, and some number (zero, one, or more) of "continuators" in the middle. (In the special case when a single pair constitutes a unit, the single pair may be called a "self-terminator".)

Fig. 3 gives a classification of kinds of units that can occur, and Fig. 4 shows a procedure for resolving any subtraction problem into its constituent units.

ERRORS

Each vertical pair of digits in a subtraction problem presents an opportunity for making an error. When an error occurs, it may be because (a) the computation the student is trying to do at that point is not the appropriate one, and/or (b) the student produces the wrong result for the computation he is trying to do. These may be referred to as "procedural errors" and "facts errors", respectively.* The remainder of this section deals with procedural errors. See Appendix II for some data on facts errors.

*Also possible are errors where the student applies the wrong algorithm altogether--where he adds, say, or subtracts "the wrong way up". Such gross errors ("algorithmic errors") are not considered further in this analysis.

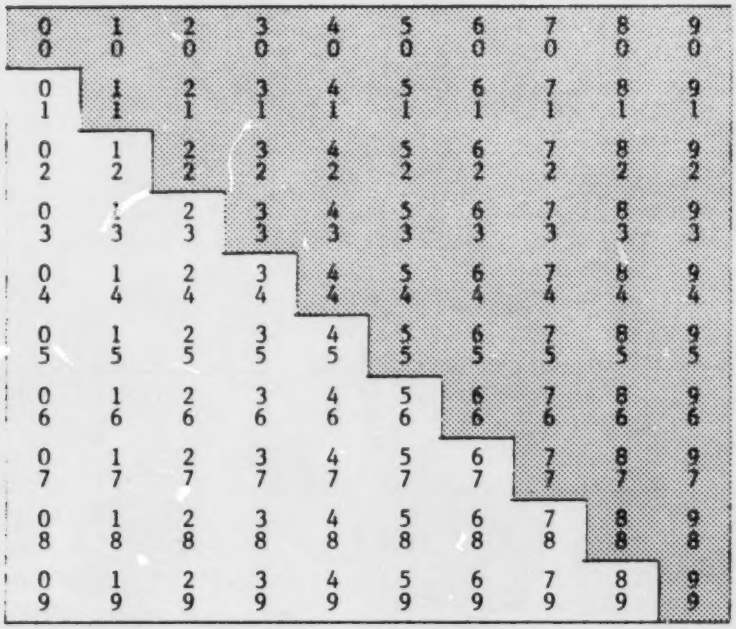

Fig. 1. When a pair begins a new unit, it is either a "self-terminator" (shaded) or an "initiator".

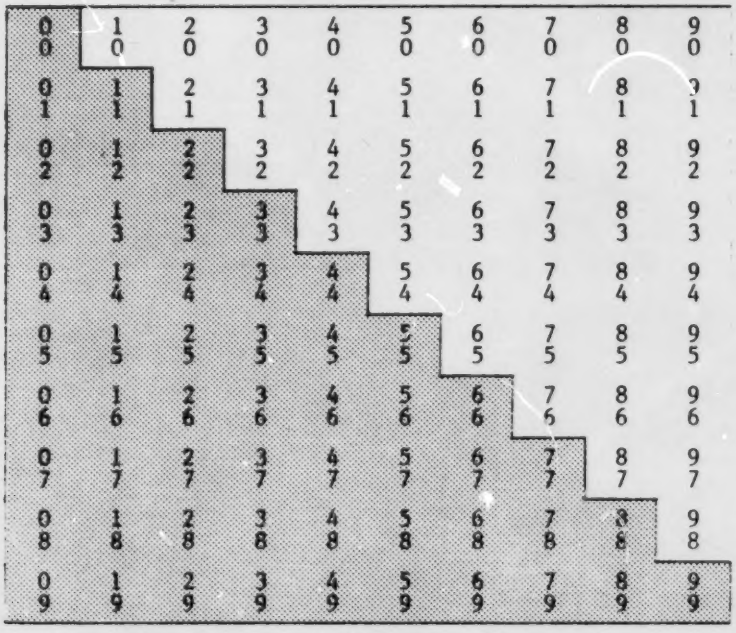

Fig. 2. When a pair is part of a unit in progress, it is either a "continuator" (shaded) or a "terminator".

4

Type	Example	Structure	Number
1-wide	$\frac{7}{-4}$	st	55
2-wide	$\frac{75}{-48}$	t - i	2025 = 45x45
3-wide	$\frac{735}{-478}$	t - c - i	111,375 = 45x55x45
4-wide	$\frac{7635}{-4678}$	t - c - c - i	6,125,625 = 45x55x55x45

Etc.

Fig. 3. Kinds of units occurring in subtraction problems. See Fig. 1 for the memberships of st and i, and Fig. 2 for the memberships of c and t.

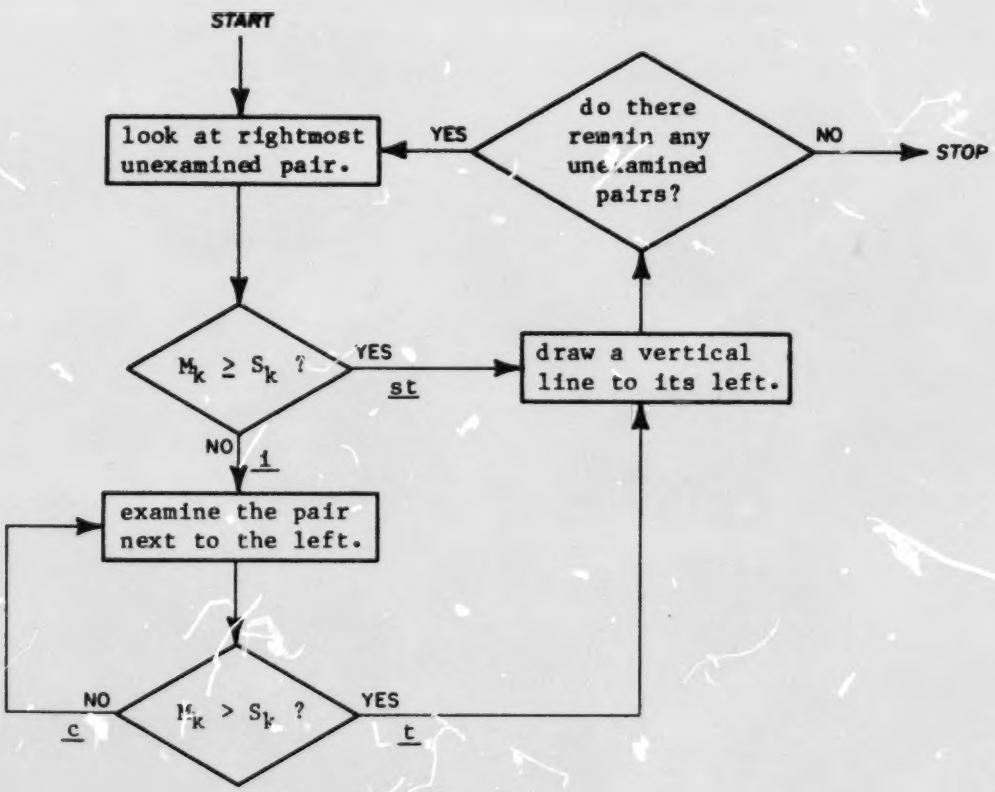

Fig. 4. Procedure for resolving a subtraction problem into its constituent units. "M_k" stands for the minuend number of the current pair, "S_k" for the subtrahend number.

The role that a particular pair has in a given problem determines the computation the student should be trying to carry out, as summarized in Fig. 5. For example, when the pair $\frac{4}{4}$ appears as a self-terminator, the student should perform neither preparatory transformation (and should produce the result 0). But when the pair $\frac{4}{4}$ appears as a continuator, he should perform both preparatory transformations (and produce the result 9).

Transformation(s) to be done before subtracting	Role of a pair			
	i	c	t	st
Increase minuend number by 10	YES	YES	NO	NO
Decrease minuend number by 1*	NO	YES	YES	NO

*Or: Increase subtrahend number by 1. (See ALGORITHMS.)

Fig. 5. Preparatory transformations to be performed on a pair as a function of its role in a subtraction problem.

As computation proceeds, the student in effect assigns a role to each successive pair and operates on it accordingly. The various possible ways he might try to do this are displayed in Fig. 6. In that figure, the cells on the diagonal cover the cases where the student assigns the correct role to a pair. The other shaded cells correspond to improbable assignments that lead to dead ends. For example, the second cell in the first row implies responding to a pair like $\frac{7}{4}$ by adding 10 to the minuend number (useless), while the first cell in the second row implies responding to a pair like $\frac{4}{7}$ by trying to subtract without any preparatory transformation (impossible). The unshaded cells show the kinds of errors a student can actually make. For example, when $\frac{7}{4}$ appears as a terminator, the student should produce the result 2; but he may mistakenly respond to the pair as a self-terminator and produce 3 instead. The t in the lower-leftmost cell in the table says that any pair which can function as a terminator is subject to that error.

Possible role assignments

Actual role of a given pair	st	i	c	t
st	all	none	h*	t=st-h
i	none	all	i	none
c	h*	i=c-h	all	none
t	t	none	none	all

*h refers to the ten "homogeneous" pairs: $\frac{0}{0}, \frac{1}{1}, \ldots \frac{9}{9}$.

Fig. 6. Possible role assignments. Each cell in the figure is associated with an actual role and a supposed assigned role. The entry in a cell tells which pairs, of those that can play the actual role, can be operated on according to the rules (Fig. 5) of the supposed assigned role.

6 Fig. 7 gives more information about these possible errors. In each case, a sample pair, the results of the correct and incorrect transformations, and the effect of the incorrect transformation, are given for illustration.

Error type	Pairs subject to error	Sample pair	After right trans.	After wrong trans.	Change in result
TSt	t	$\frac{7}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	+1
StT	t	$\frac{7}{6}$	$\frac{7}{6}$	$\frac{6}{6}$	-1
CSt	h*	$\frac{5}{5}$	$\frac{14}{5}$	$\frac{5}{5}$	9-0
StC	h	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{14}{5}$	0-9
CI	i	$\frac{3}{4}$	$\frac{12}{4}$	$\frac{13}{4}$	+1
IC	i	$\frac{3}{4}$	$\frac{13}{4}$	$\frac{12}{4}$	-1

*h refers to the ten "homogeneous" pairs: $\frac{0}{0}, \frac{1}{1}, \ldots \frac{9}{9}$.

Fig. 7. Possible procedural errors. "TSt" means that a terminator is mistakenly treated like a self-terminator (the lower left-most cell in Fig. 6). "9-0" means that a result of 9 is turned into a result of 0.

Single-unit problems can easily be constructed that provide opportunities for errors of types TSt, CSt, and CI. However, only multiple-unit problems can provide realistic opportunities for errors of types StT, StC, and IC.

SECONDARY ERRORS

When deciding how to operate on a pair, the primary cue available to the student is the way he operated on the previous pair. The role of the previous pair determines whether the current pair begins a new unit or is part of a unit in progress, and that information is sufficient for the student to classify and respond to the current pair. Fig. 8 summarizes the cue relationships. In words: If the previous pair is a self-terminator or a terminator, then the current pair begins a new unit and is either a self-terminator or an initiator (as determined by Fig. 1). If the previous pair is an initiator or a continuator, then the current pair is part of a unit in progress and is either a continuator or a terminator (as determined by Fig. 2).

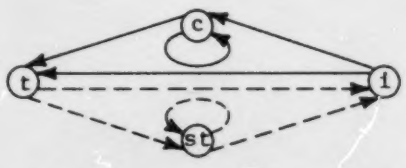

Fig. 8. Valid role sequences for successive pairs in subtraction problems. The solid arrows represent within-unit sequences, while the dashed arrows represent unit-boundary sequences. For example, the lower right-hand arrow says that a self-terminator may be followed by an initiator.

There are two ways a student can go wrong in assigning a role to a pair. He can treat the current pair incorrectly in spite of having treated the previous pair correctly. Or he can treat the current pair incorrectly because of having treated the previous pair incorrectly. These can be referred to as primary and secondary errors, respectively.

Examination shows that secondary errors can follow upon only two kinds of procedural errors, CSt and StC. (In the other four cases, the same cue is provided when an error is made as when it is not.) Since errors CSt and StC are possible only for homogeneous pairs, these secondary errors will be easy to see--the expected nine/zero under a homogeneous pair will appear as a zero/nine, and the digit under the (non-h) pair just to the left will be too large/too small by one.

ALGORITHMS

In order to design subtraction tests and exercises, it is necessary to consider, not only the kinds of problems that are possible, but also the different approaches that students may bring to solving them.

In approaching a simple subtraction problem, one can think either like this:

8 less 5 gives _

or like this:

5 plus _ gives 8

In dealing with more complicated problems, using either approach, one can deal with pairs like $\frac{4}{7}$ either by "borrowing" or by "carrying" (these words are used for convenience; persons using the algorithms may be thinking in other terms altogether).

The four resulting algorithms for subtraction are illustrated in Fig. 9 below.

SUBTRACT/BORROW		SUBTRACT/CARRY	
6 14 7 ~~5~~ 6 − 4 9 7 2 5 9	16 − 7 = _ 14 − 9 = _ 6 − 4 = _	7 5 6 5 10 − 4 ~~9~~ 7 2 5 9	16 − 7 = _ 15 − 10 = _ 7 − 5 = _
ADD/BORROW		**ADD/CARRY**	
6 14 7 ~~5~~ 6 − 4 9 7 2 5 9	7 + _ = 16 9 + _ = 14 4 + _ = 6	7 5 6 5 10 − ~~4~~ ~~9~~ 7 2 5 9	7 + _ = 16 10 + _ = 15 5 + _ = 7

Fig. 9. Four algorithms for subtraction. Equations showing the thinking process are given for each case, along with sample written work as it might be produced by a meticulous student.

8 Inasmuch as the present analysis is mainly a logical one, the distribution of usage of the described algorithms is not known (and the existence/usage of other algorithms is not ruled out).

A prominent difference between BORROW methods and CARRY methods is illustrated in the following comparison.

```
  8 9 9                    9 0 0 8
  9 0 0 8                    5
- 6 5 4 9                - 6 5 4 9
-------                  -------
      9                        9
```

In the example on the left, it is necessary, strictly speaking, to perform all the transformations shown before computing can begin. This "zero exception" in BORROWing occurs whenever zero is the minuend number in a continuator.

Logically, a corresponding exception occurs in CARRYing when nine is the subtrahend number in a continuator. However, in that case the exception expresses itself merely as the transitory appearance of a two-digit number (10) as the subtrahend number of a pair, and no propagating transformation is entailed (see the CARRY written work in Fig. 9). Both cases should be covered in problem sets.

An outcome of the analysis in this section is a recommendation concerning a preferred algorithm for subtraction. Of the four methods described, ADD/CARRY appears to have two special advantages. (a) As an ADD method, it uses directly the already-familiar "addition facts", providing extra reinforcement for these and avoiding the need to learn a competing set of "subtraction facts". (b) As a CARRY method, it avoids the "zero exception" involved in BORROWing and the special teaching that is probably necessary to deal with it. (In CARRYing, the situation is so simple the student may never even need to be aware that a special case exists.)

The advantage of ADD/CARRY can be maximized by actually writing subtraction problems in addition format. Eg,

```
    5
  6 5 4 9
+ [   9 ]
---------
  9 0 0 8
```

This approach, in effect, absorbs subtraction into addition --all the familiar cues of addition are brought to bear to the student's advantage. And the divorce from conventional methods of subtraction is so complete that the approach probably can be learned as easily by remedial students as by beginners.

To handle situations like long division, checkbooks, and standardized tests, it is necessary to make the transition from addition format to ADD/CARRY format, but compared to the usual methods of teaching subtraction, this should turn out to be relatively straightforward.

A 3-wide subtraction problem has four possible unit compositions--a 3-wide unit standing alone, a 2-wide unit followed by a 1-wide unit, a 1-wide unit followed by a 2-wide unit, and three 1-wide units. The following list gives similar breakdowns in abbreviated form for all problem sizes from 1-wide through 7-wide. Extending the list to problems of any desired size is a straightforward, if somewhat tedious, task.

1
- - - - -
2
11
- - - - -
3
21,12
111
- - - - -
4
31,13
22
211,121,112
1111
- - - - -
5
41,14
32,23
311,131,113
221,212,122
2111,1211,1121,1112
11111
- - - - -
6
51,15
42,24
411,141,114
33
321,312,231,213,132,123
3111,1311,1131,1113
222
2211,2121,2112,1221,1212,1122
21111,12111,11211,11121,11112
111111
- - - - -
7
61,16
52,25
511,151,115
43,34
421,412,241,214,142,124
4111,1411,1141,1114
331,313,133
322,232,223
3211,3121,3112,2311,2131,2113,1321,1312,1231,1213,1132,1123
31111,13111,11311,11131,11113
2221,2212,2122,1222
22111,21211,21121,21112,12211,12121,12112,11221,11212,11122
211111,121111,112111,111211,111121,111112
1111111

APPENDIX II--DIFFICULTY OF SUBTRACTION FACTS

The data presented here were collected in the mid-1960's from a sample of 700+ seventh grade students from inner city schools from all parts of the country. Students had five minutes to respond to the 100 "facts". The test had two pages, each containing five rows of 10 problems each. Problems were distributed among the rows in a way intended to minimize inter-problem cueing and to render the rows approximately equal in difficulty. Five different forms of the test were prepared by rotating the five rows on each sheet; rotating the two sheets increased the number of different forms to ten. The tests were systematically interleaved before shipment so as to distribute the ten forms evenly across classes and students. When the tests were scored, omitted items at the end of a test were ignored. These procedures were designed to minimize bias due to order of the items, and to distribute residual "noise" nearly evenly across all the items. The regularity exhibited by the data indicates that these goals were substantially realized. Generalizability of the data to other populations is risky with respect to absolute difficulty, but probably safe enough with respect to relative difficulty of groups of items. Only the "zero facts" in the first row and first column are judged to be open to large changes in relative difficulty from one student population to another.

Subtrahend number

Answer	0	1	2	3	4	5	6	7	8	9
0	1	5	4	5	6	4	4	4	5	4
1	15	4	3	3	4	2	4	3	3	4
2	10	5	3	4	4	4	6	6	4	10
3	12	4	2	3	5	5	5	7	11	8
4	12	5	7	4	4	5	4	11	9	10
5	14	3	4	5	5	3	8	10	11	10
6	12	6	5	7	5	7	5	12	10	10
7	13	4	5	7	10	8	11	6	11	14
8	12	5	6	10	8	14	12	10	6	12
9	12	4	8	11	11	11	13	13	9	7

Fig. 10. Difficulty of the one hundred "basic facts" of subtraction. Each entry shows the percentage of students failing to give the correct answer to the corresponding "basic fact". Entries whose values are 8% or greater are shaded.

The arrangement used in Fig. 10 displays well the order in the data. In constructing problem sets, however, it is more convenient to have the data arranged as in Fig 11.

Minuend number

Subtrahend number	0	1	2	3	4	5	6	7	8	9
0	1	15	10	12	12	14	12	13	12	12
1	4	5	4	5	4	5	3	6	4	5
2	6	8	4	3	3	2	7	4	5	5
3	7	10	11	5	3	4	3	4	5	7
4	5	10	8	11	6	4	4	5	4	5
5	3	7	8	14	11	4	2	4	5	5
6	4	8	5	11	12	13	4	4	6	5
7	7	11	10	12	6	10	13	4	3	6
8	4	11	9	11	10	11	6	9	5	3
9	4	10	8	10	10	10	14	12	7	4

Fig. 11. Subtraction "facts" difficulty data from Fig. 10 rearranged to correspond cell-by-cell to Fig. 1.

APPENDIX III--A LISTING OF ALL POSSIBLE 2-WIDE UNITS

Every unit larger than a 1-wide must have the same initiator and terminator as some 2-wide unit. (To see that this is true, note that any larger unit becomes a 2-wide if its continuators are dropped out, and they always can be.) Hence, it is useful to have available an overview of the full population of 2-wide units.

The full population of 2-wide expressions contains 10,000 members (from $\frac{00}{00}$ through $\frac{99}{99}$). Fig. 12 classifies these according to their structures. The last line of the figure describes the full population of 2-wide units.

Structure	Example	Number	Nature
st - st	$\frac{45}{44}$	55x55=3025	Double 1-wide's
i - st	$\frac{45}{54}$	45x55=2475	Not valid subtractions
c - i	$\frac{44}{45}$	55x45=2475	Not valid subtractions
t - i	$\frac{54}{45}$	45x45=2025	2-wide units

Fig. 12. Classification of 2-wide expressions. Consult Fig. 1 for the memberships of st and i, and Fig. 2 for the memberships of c and t.

Fig. 13 shows that these 2025 units fall into 81 classes according to correct answer, and that the size of these classes varies from a low of 1 to a high of 81. (Each class is generated by pairing a diagonal of initiators from Fig. 1 with a diagonal of terminators from Fig. 2.) The remainder of the appendix comprises an explicit listing of the 2025 units grouped according to these 81 classes.

1 9	11 8	21 7	31 6	41 5	51 4	61 3	71 2	81 1
2 18	12 16	22 14	32 12	42 10	52 8	62 6	72 4	82 2
3 27	13 24	23 21	33 18	43 15	53 12	63 9	73 6	83 3
4 36	14 32	24 28	34 24	44 20	54 16	64 12	74 8	84 4
5 45	15 40	25 35	35 30	45 25	55 20	65 15	75 10	85 5
6 54	16 48	26 42	36 36	46 30	56 24	66 18	76 12	86 6
7 63	17 56	27 49	37 42	47 35	57 28	67 21	77 14	87 7
8 72	18 64	28 56	38 48	48 40	58 32	68 24	78 16	88 8
9 81	19 72	29 63	39 54	49 45	59 36	69 27	79 18	89 9

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Fig. 13. Structure of the full population of 2-wide units. Looking at the top row, the first entry says there are nine units whose correct answer is 1, eight units whose correct answer is 11, etc. The major divisions of the figure correspond to pages of the explicit listing which follows.

10	20	30	40	50	60	70	80	90
9	19	29	39	49	59	69	79	89

10	20	30	40	50	60	70	80	90
8	18	28	38	48	58	68	78	88
11	21	31	41	51	61	71	81	91
9	19	29	39	49	59	69	79	89

10	20	30	40	50	60	70	80	90
7	17	27	37	47	57	67	77	87
11	21	31	41	51	61	71	81	91
8	18	28	38	48	58	68	78	88
12	22	32	42	52	62	72	82	92
9	19	29	39	49	59	69	79	89

10	20	30	40	50	60	70	80	90
6	16	26	36	46	56	66	76	86
11	21	31	41	51	61	71	81	91
7	17	27	37	47	57	67	77	87
12	22	32	42	52	62	72	82	92
8	18	28	38	48	58	68	78	88
13	23	33	43	53	63	73	83	93
9	19	29	39	49	59	69	79	89

20	30	40	50	60	70	80	90	
9	19	29	39	49	59	69	79	

20	30	40	50	60	70	80	90	
8	18	28	38	48	58	68	78	
21	31	41	51	61	71	81	91	
9	19	29	39	49	59	69	79	

20	30	40	50	60	70	80	90	
7	17	27	37	47	57	67	77	
21	31	41	51	61	71	81	91	
8	18	28	38	48	58	68	78	
22	32	42	52	62	72	82	92	
9	19	29	39	49	59	69	79	

20	30	40	50	60	70	80	90	
6	16	26	36	46	56	66	76	
21	31	41	51	61	71	81	91	
7	17	27	37	47	57	67	77	
22	32	42	52	62	72	82	92	
8	18	28	38	48	58	68	78	
23	33	43	53	63	73	83	93	
9	19	29	39	49	59	69	79	

30	40	50	60	70	80	90		
9	19	29	39	49	59	69		

30	40	50	60	70	80	90		
8	18	28	38	48	58	68		
31	41	51	61	71	81	91		
9	19	29	39	49	59	69		

30	40	50	60	70	80	90		
7	17	27	37	47	57	67		
31	41	51	61	71	81	91		
8	18	28	38	48	58	68		
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